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Maximal and Minimal α -Open Sets

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Abstract.
In 2001 and 2003, Nakaok and Oda [3] and [4] introduced the notation of maximal open sets and minimal open sets in topological spaces. In 2010, Miguel, Saeid and Seithuti [2], introduced the notation of maximal θ-open sets and minimal θ -closed sets in topological spaces. In this paper, we introduce new classes of sets called maximal α-open sets and minimal α -open sets and investigate some of their fundamental properties.
Key word and phrases: θ-open, maximal open sets, minimal closed, maximal θ-open sets and minimal θ – closed.

I. Introduction.

The study of α -open sets and their properties were initiated by Njastad [5] in 1965; his introduction of α - open sets. Andrijevic [1] gave some properties of α - closure of a set A is denoted by αCl(A), and defined as intersection of all α -closed sets containing the set A.

F. Nakaok and N. Oda [3] and [4] introduced the notation of maximal open sets and minimal open sets in topological spaces. In (2010) Mlguel Caldas, Saeid Jafari and Seithuti P. Moshokes [2]; introduce the notion of maximal θ-open, minimal θ-closed, θ-semi maximal open and θ-semi minimal closed and investigate some of the fundamental properties.

The purpose of the present paper is to introduce the concept of a new class of open sets called maximal α - open sets and minimal α -closed sets. We also investigate some of their fundamental properties.

II. Definition and Preliminaries.

Definition2.1. [5] A subset *A* of a space *X* is said to be α-open set if $A \subseteq Int(Cl(Int(A)))$. The complement of all α-open set is said to be α-closed. As in the usual sense, the intersection of all α-closed sets of *X* containing *A* is called the α-closure of *A.* also the union of all α-open sets of *X* contained in *A* is called the α-interior of *A*.

Definition2.2. [6] A subset *A* of a space *X* is said to be θ-open set if for each $x \in A$, there exists an open set *G* such that $x \in G \subseteq Cl(G) \subseteq A$.

Definition2.3. [4] A proper nonempty open set *U* of *X* is said to be a maximal open set if any open set which contains *U* is *X* or *U*.

Definition2.4. [4] A proper nonempty open set *U* of *X* is said to be a minimal open set if any open set which contained in *U* is ϕ or *U*.

Definition2.5. [3] A proper nonempty closed subset *F* of *X* is said to be a maximal closed set if any closed set which contains *F* is *X* or *F*.

Definition2.6. [3] A proper nonempty closed subset *F* of *X* is said to be a minimal closed set if any closed set which contained in *F* is ϕ or *F*.

Definition2.7. [2] A proper nonempty θ-open set *U* of *X* is said to be a maximal θ-open set if any θ-open set which contains *U* is *X* or *U*.

Definition2.8. [2] A proper nonempty θ-closed set *B* of *X* is said to be a minimal θ-closed set if any θ-closed set which contained in *B* is ϕ or *F*.

III. Maximal and minimal α-open sets.

Definition3.1. A proper nonempty α-open set *A* of *X* is said to be a maximal α-open set if any α-open set which contains *A* is *X* or *A*.

Definition3.2. A proper nonempty α-closed set *B* of *X* is said to be a minimal α-closed set if any α-closed set which contained in *B* is ϕ or *B*.

The family of all maximal α-open (resp.; minimal α-closed) sets will be denoted by $M_a\alpha O(X)$ (resp.; $M_i\alpha C(X)$). We set
$M_a\alpha O(X,x) =\{ A: x \in A \in M_a\alpha O(X)\}$, and
$M_i\alpha C(X,x) =\{ F: x \in A \in M_i\alpha C(X)\}$.

Theorem3.3. Let A be a proper nonempty subset of X. Then A is a maximal α-open set if $X\backslash A$ is a minimal α-closed set.

Proof. Necessity. Let A be a maximal α-open. Then $A \subset X$ or $A \subset A$. Hence, $\phi \subset X\backslash A$ or $X\backslash A \subset X\backslash A$. Therefore, by Definition 3.2, $X\backslash A$ is a minimal α-closed set.

Sufficiency. Let $X\backslash A$ is be a minimal α-closed set. Then $\phi \subset X\backslash A$ or $X\backslash A \subset X\backslash A$. Hence, $A \subset X$ or $A \subset A$ which implies that A is a maximal α-open set.

The following example shows that maximal-open sets and maximal α-open sets are in general independent.

Example3.4. Consider $X = \{a, b, c\}$ with topology $\tau = \{\phi, X, \{a\}\}$. Then the family of αO(X) $= \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\}$. So $\{a\}$ is a maximal open in X, which is not maximal α-open in X, and $\{a, b\}$ is a maximal α-open in X, which is not maximal open in X.

Theorem3.5. Any open set if it is a maximal α-open set then it is a maximal open set.

Proof. Let U be open and maximal α-open set in a topological space X. We want to prove that U is a maximal open set. Suppose that U is not maximal open set, then $U \neq X$ and there exists an open set G such that $U \subset G$ and $U \neq G$, but every open set is α-open, this implies that G is a α-open set containing U and $U \neq G$ and $U \neq X$, which is contradiction. Hence U is a maximal open set.

Theorem3.6. For any topological space X, the following statements are true.

1) Let A be a maximal α-open set and B be a α-open. Then $A \cup B = X$ or $B \subset A$.
2) Let A and B be maximal α-open sets. Then $A \cup B = X$ or $B = A$.
3) Let F be a minimal α-closed set and G be a α-closed set. Then $F \cap G = \phi$ or $F \subset G$.
4) Let F and G be minimal α-closed sets. Then $F \cap G = \phi$ or $F = G$.

Proof (1). Let A be a maximal α-open set and B be a α-open set. If $A \cup B = X$, then we are done. But if $A\cup B \neq X$, then we have to prove that $B \subset A$, but $A \cup B \neq X$ means
$B \subset A \cup B$ and $A \subset A \cup B$. Therefore we have $A \subset A \cup B$ and A is a maximal α-open, then by Definition 3.1, $A \cup B = X$ or $A \cup B = A$, but $A \cup B \neq X$, then $A \cup B = A$, which implies $B \subset A$.

Proof (2). Let A and B be maximal α-open sets. If $A \cup B = X$, then we have done. But if $A \cup B \neq X$, then we have to prove that $B = A$. Now $A \cup B \neq X$, means $B \subset A \cup B$ and $A \subset$ A $\cup B$. Now $A \subset A \cup B$ and A is a maximal α-open, then by Definition 3.1, $A \cup B = X$ or
$A \cup B = A$, but $A \cup B \neq X$, therefore, $A \cup B = A$, which implies $B \subset A$. Similarly if $B \subset A \cup B$ we obtain $A \subset B$. Therefore $B = A$.

Proof (3). Let F be a minimal α-closed set and G be a α-closed set. If $F \cap G =\phi$, then there is nothing to prove. But if $F \cap G \neq \phi$, then we have to prove that $F \subset G$. Now if
$F \cap G \neq \phi$, then $F \cap G \subset F$ and $F \cap G \subset G$. Since $F \cap G \subset F$ and given that F is minimal α-closed, then Definition 3.2, $F \cap G = F$ or $F \cap G =\phi$. But $F \cap G \neq\phi$ then
$F \cap G = F$ which implies $F \subset G$.

Proof (4). Let F and G be two minimal α-closed sets. If $F \cap G =\phi$, then there is nothing to prove. But if $F \cap G\neq \phi$, then we have to prove that $F = G$. Now if $F \cap G\neq \phi$, then $F \cap G \subset F$ and $F \cap G \subset G$. Since $F \cap G \subset F$ and given that F is minimal α-closed, then by Definition 3.2, $F \cap G = F$ or $F \cap G=\phi$. But $F \cap G \neq \phi$ then $F \cap G = F$ which implies $F \subset G$. Similarly if $F \cap G \subset G$ and given that G is minimal α-closed, then by Definition 3.2, $F \cap G = G$ or $F \cap G =\phi$. But $F \cap G\neq \phi$, then $F \cap G = G$ which implies $G \subset F$. Then $F=G$.

Theorem3.7.

1) Let A be a maximal α-open set and x is an element of $X\backslash A$. Then $X\backslash A \subset B$ for any α-open set B containing x.
2) Let A be a maximal α-open set. Then either of the following (i) or (ii) holds:
(i) For each $x \in X\backslash A$, and each α-open set B containing x, $B=X$.
(ii) There exists a α-open set B such that $X\backslash A \subset B$, and $B \subset X$.
3) Let A be a maximal α-open set. Then either of the following (i) or (ii) holds:

(i) For each $x \in X\backslash A$, and each α-open set B containing x, we have $X\backslash A \subset B$.
(ii) There exists a α-open set B such that $X\backslash A = B \neq X$.

Proof. 1) Since $x \in X\backslash A$, we have $B \not\subset A$ for any α-open set B containing x. Then $A \cup B = X$, by Theorem 3.5(1). Therefore, $X\backslash A \subset B$.

2) If (i) dose not hold, then there exists an element x of $X\backslash A$, and a α-open set B containing x such that $B \subset X$. By (1) we have $X\backslash A \subset B$.

3) If (ii) dose not hold, then we have $X\backslash A \subset B$ for each $x \in X\backslash A$ and each α-open set B containing x. Hence, we have $X\backslash A \subset B$.

Theorem3.8. Let A, B and C be maximal α-open sets such that $A \neq B$. If $A \cap B \subset C$, then either $A=C$ or $B=C$.

Proof. Given $A \cap B \subset C$. If $A= C$, then there is nothing to prove. But If $A \neq C$, then we have to prove $B= C$. Using Theorem 3.6(2), we have

$$
\begin{aligned}
B \cap C &= B \cap [C \cap X] = B \cap [C \cap (A \cup B)] \\
&= B \cap [(C \cap A) \cup (C \cap B)] \\
&= (B \cap C \cap A) \cup (B \cap C \cap B) \\
&= (A \cap B) \cup (C \cap B), \text{ since } A \cap B \subset C \\
&= (A \cup C) \cap B \\
&= X \cap B = B, \text{ since } A \cup C = X.
\end{aligned}
$$

This implies $B \subset C$ also from the definition of maximal α-open set it follows that $B=C$.

Theorem3.9. Let A, B and C be maximal α-open sets which are different from each ether. Then $(A \cap B) \not\subset (A \cap C)$.

Proof. Let $(A \cap B) \subset (A \cap C)$. Then $(A \cap B) \cup (C \cap B) \subset (A \cap C) \cup (C \cap B)$. Hence, $(A \cup C) \cap B \subset C \cap (A \cup B)$. Since by Theorem 3.6(2), $A \cup C = X$. We have $X \cap B \subset C \cap X$ which implies $B \subset C$. From the definition of maximal α-open set it follows that $B = C$. Contradiction to the fact that A, B and C are different from each other. Therefore $(A \cap B) \not\subset (A \cap C)$.

Theorem3.10. (1) Let F be a minimal α-closed set of X. If $x \in F$, then $F \subset G$ for any α-closed set G containing x.

(2) Let F be a minimal α-closed set of X. Then $F = \cap\{G: G \in \alpha C(X)\}$.

Proof. (1). Let $F \in M_i\alpha C(X, x)$ and $G \in \alpha C(X, x)$, such that $F \not\subset G$. This implies that $F \cap G \subset F$ and $F \cap G \neq \phi$. But since F is minimal α-closed, by Definition 2.2, $F \cap G = F$ which contradicts the relation that $F \cap G \subset F$. Therefore $F \subset G$.

(2) By (1) and the fact that F is α-closed containing x, we have $F \subset \cap\{G: G \in \alpha C(X, x)\} \subset F$. Therefore we have the result.

Theorem3.11. (1) Let F and $\{F_\lambda\}_{\lambda \in \Lambda}$ be minimal α-closed sets. If $F \subset \cup_{\lambda \in \Lambda} F_\lambda$, then there exists $\lambda \in \Lambda$ such that $F=F_\lambda$.

(2) Let F and $\{F_\lambda\}_{\lambda \in \Lambda}$ be minimal α-closed sets. If $F \neq F_\lambda$ for each $\lambda \in \Lambda$, then $\cup_{\lambda \in \Lambda} F_\lambda \cap F = \phi$.

Proof. (1) Let F and $\{F_\lambda\}_{\lambda \in \Lambda}$ be minimal α-closed sets with $F \subset \cup_{\lambda \in \Lambda} F_\lambda$ .we have to prove that $F \cap F_\lambda \neq \phi$. Since if $F \cap F_\lambda = \phi$, then $F_\lambda \subset X \setminus F$ and hence, $F \subset \cup_{\lambda \in \Lambda} F_\lambda \subset X \setminus F$ which is a contradiction. Now as $F \cap F_\lambda \neq \phi$, then $F \cap F_\lambda \subset F$ and $F \cap F_\lambda \subset F_\lambda$. Since $F \cap F_\lambda \subset F$ and give that F is minimal α-closed, then by Definition2.1, $F \cap F_\lambda = F$ or $F \cap F_\lambda = \phi$. But $F \cap F_\lambda \neq \phi$, then $F \cap F_\lambda = F$ which implies $F \subset F_\lambda$ .Similarly $F \cap F_\lambda \subset F_\lambda$ and give that F_λ is minimal α-closed, then by Definintion2.1, $F \cap F_\lambda = F_\lambda$ or $F \cap F_\lambda = \phi$. But $F \cap F_\lambda \neq \phi$ then $F \cap F_\lambda = F_\lambda$ which implies $F_\lambda \subset F$. Therefore, $F= F_\lambda$.

(2) Suppose that $(\cup_{\lambda \in \Lambda} F_\lambda) \cap F \neq \phi$ then there exists $\lambda \in \Lambda$ such that $F_\lambda \cap F \neq \phi$. By Theorem 3.6(4), we have $F= F_\lambda$ which is a contradiction to the fact $F \neq F_\lambda$. Hence,
$(\cup_{\lambda \in \Lambda} F_\lambda) \cap F= \phi$.

Some Properties of Maximal -open set.

Theorem3.12. Let U be a maximal α-open set. Then $\alpha Cl(U)=X$ or $\alpha Cl(U)=U$.

Proof. Since U is a maximal α-open set, the only following cases (1) and (2) occur by Theorem 3.7(2):

(1) For each $x \in X\backslash U$ and each α-open set W of x, we have $X\backslash U \subset W$, let x be any element of $X\backslash U$ and W be any α-open set of x. Since $X\backslash U \neq W$, we have $W \cap U \neq \phi$ for any α-open set W of x. Hence, $X\backslash U \subset \alpha Cl(U)$. Since $X=X \cup (X\backslash U) \subset U \cup \alpha Cl(U) = \alpha Cl(U) \subset X$, we have $\alpha Cl(U)= X$.

(2) There exists a α-open set W such that $X\backslash U=W\neq X$, since $X\backslash U = W$ is a α-open set, U is a α-closed set. Therefore, $U= \alpha Cl(U)$.

Theorem3.13. Let U be a maximal α-open set. Then $\alpha Int(X\backslash U) =X\text{-}U$ or $\alpha Int(X\backslash U)= \phi$.

Proof. By Theorem3.7, we have following cases (1) $\alpha Int(X\backslash U) = \phi$ or (2) $\alpha Int(X\backslash U) = X\backslash U$.

Theorem3.14. Let U be a maximal α-open set and S a nonempty subset of $X\backslash U$. Then $\alpha Cl(S)= X\backslash U$.

Proof. Since $\phi\neq S \subset X\backslash U$, we have $W \cap S \neq\phi$ for any element x of $X\backslash U$ and any α-open set W of x by Theorem 3.12. Then $X\backslash U \subset \alpha Cl(U)$. Since $X\backslash U$ is a α-closed set and $S \subset X\backslash U$, we see that $\alpha CL(S) \subset \alpha Cl(X\backslash U) = X\backslash U$. Therefore $X\backslash U= \alpha Cl(S)$.

Corollary 3.15. Let U be a maximal α-open set and M a subset of X with $U\subset M$. Then $\alpha Cl(M) =X$.

Proof. Since $U\subset M \subset X$, there exists a nonempty subset S of $X\backslash U$ such that $M= U\cup S$. Hence we have $\alpha Cl(M)= \alpha Cl(U\cup S)= \alpha Cl(U) \cup \alpha Cl(S) \supset (X\backslash U) \cup U= X$ by Theorem 3.14. Therefore $\alpha Cl(M) =X$.

Theorem3.16. Let U be a maximal α-open set and assume that the subset $X\backslash U$ has two element at least. Then $\alpha Cl(X\backslash\{a\}) =X$ for any element of $X\backslash U$.

Proof. Since $U \subset X\backslash\{a\}$ by our assumption, we have the result by Corollary 3.15.

Theorem3.17. Let U be a maximal α-open set, and N be a proper subset of X with $U \subset N$. Then, $\alpha Int(N)= U$.

Proof. If $N = U$, then $\alpha Int(N)= \alpha Int(U)=U$. Otherwise, $N\neq U$, and hence $U \subset N$. It follows that $U \subset \alpha Int(N)$. Since U is a maximal α-open set, we have also $\alpha Int(N) \subset U$. Therefore $\alpha Int(N) = U$.

References

[1.] D. Andrijevic. Some properties of the topology of α-sets, Math. Vesnik, 36 (1984), 1-10.
[2.] C. Miguel, J. Saeid and M. Seithuti. On some new maximal and minimal sets via θ- open sets.Commun. Korean Math. Soc. 25 (2010), No.4, 623-628.
[3.] F. Nakaok and N. Oda. Some application of minimal open sets, Int. J. Math. Math. Sci. 27 (2001), no. 8, 471-476.
[4.] --------, Some Properties of maximal open sets, Int. J. Math. Math. Sci. (2003), no. 21, 1331-1340.
[5.] O. Njastad, On some classes of nearly open sets, Pacific J. Math., 15(3) (1965), 961-970.
[6.] N. Velicko. H-closed topological spaces, Amer. Math. Soc. Transl.,78 (2) (1968), 103- 118.